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Electron Self-Energy, Vacuum Polarization, and Vertex Correction

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In the framework of the finite quantum electrodynamics developed in the previous paper, electron self-energy, vacuum polarization, and vertex correction are calculated. It has turned out that the electron–neutrino mass difference can be reproduced in a model where this mass difference is of pure electromagnetic origin. A positive sign of proton–neutron mass difference is obtained within our present theory. Furthermore, it is shown that our theory can give a clue to overcome the possible crisis of QED arising from the recent report of the discrepancy between theoretical and experimental values for the muon anomalous magnetic moment as an evidence for a possible breakdown of QED.

1. INTRODUCTION

In the previous paper (Cheon, 1978), we developed a finite theory of quantum electrodynamics which contains the fundamental length. Our basic equation of motion for a boson is the Bopp equation, which is a differential equation of fourth order. It is, of course, Lorentz invariant, while the equation of motion for a fermion is a quadratic differential equation. The propagators obtained from these equations have the same forms as those given by Feynman (1949), i.e., the particle propagator subtracted by that of the tildon which is confined only in the virtual state by the mass relation as a result of reflection of the discrete space-time associated with the fundamental length.

It will be shown that our theory gives finite results for electron self-energy, vacuum polarization, and vertex correction.

As is well known, the renormalization method can yield beautiful agreement between theory and experiment, in particular, anomalous magnetic moments of the electron and the muon. However, the recent experiment (Bailey et al., 1975) shows a possible discrepancy between theoretical and

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measured values for the $(g - 2)$ factor anomaly of the muon. One may also expect a breakdown of QED at somewhat high energy. We will present a discussion on how our theory is powerful enough to overcome a possible crisis of QED.

Since the self-energy can be calculated in the framework of our theory, a possibility will be expected to reproduce a correct sign for the proton–neutron mass difference. Furthermore, we propose here a model in which the electron consists of a neutrino and a charge.

The metric used in the present paper is $g_{00} = -g_{11} = -g_{22} = -g_{33} = 1$, and we use the natural unit $\hbar = c = 1$.

2. ELECTRON SELF-ENERGY

The propagators of a photon and an electron are given in the previous paper in the forms

$$\frac{1}{(1 - l_0{}^2k^2)k^2} = \frac{1}{k^2} - \frac{1}{k^2 - 1/l_0{}^2} \tag{2.1}$$

$$\frac{\hbar/l_0}{(\not{p} + \tilde{m}_+c)(\not{p} - \tilde{m}_-c)} = \frac{\hbar/l_0c}{\tilde{m}_+ + \tilde{m}_-}\left(\frac{1}{\not{p} - \tilde{m}_-c} - \frac{1}{\not{p} + \tilde{m}_+c}\right) \tag{2.2}$$

where the masses $\tilde{m}_\pm$ are expressed with the fundamental length l_0 and the bare mass m_0 as

$$\tilde{m}_\pm{}^2 = \frac{1}{2}\left(\frac{\hbar}{l_0c}\right)^2\{1 \pm [1 - (2l_0m_0c/\hbar)^2]^{1/2}\} \tag{2.3}$$

However, since the fundamental length in the photon propagator might be different from that in the electron (more generally fermion) propagator, we express them here as follows:

$$\frac{1}{k^2} - \frac{1}{k^2 - \Lambda^2} \tag{2.4}$$

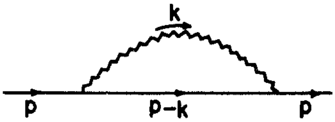
$$\frac{(\tilde{m}_+{}^2 + \tilde{m}_-{}^2)^{1/2}}{\tilde{m}_+ + \tilde{m}_-}\left(\frac{1}{\not{p} - \tilde{m}_-} - \frac{1}{\not{p} + \tilde{m}_+}\right) \tag{2.5}$$

(Hereafter we use the natural unit $\hbar = c = 1$.)

The lowest order of the electron self-energy (see Figure 1) is given in the usual representation, i.e.,

$$\Delta m = \frac{4\pi\alpha}{2i\tilde{m}_-(2\pi)^4}(\bar{u}, \Sigma u) \tag{2.6}$$

Fig. 1. Feynman graph of the lowest order of the electron self-energy.



where α is the fine structure constant, u is the wave function of the free electron which is normalized as $(\tilde{u}, u) = 2\tilde{m}_-$, and

$$\Sigma = \int \gamma_\mu \frac{(\tilde{m}_+{}^2 + \tilde{m}_-{}^2)^{1/2}}{(\not{p} - \not{k} + \tilde{m}_+)(\not{p} - \not{k} - \tilde{m}_-)} \gamma^\mu \frac{-\Lambda^2}{k^2(k^2 - \Lambda^2)} d^4k \qquad (2.7)$$

As a method of calculation of this integral is shown in Feynman's paper (1949), we follow it here by rewriting

$$\Sigma = \frac{(\tilde{m}_+{}^2 + \tilde{m}_-{}^2)^{1/2}}{\tilde{m}_+ + \tilde{m}_-}$$

$$\times \int \gamma_\mu \left[\frac{\not{p} - \not{k} + \tilde{m}_-}{(p-k)^2 - \tilde{m}_-{}^2} - \frac{\not{p} - \not{k} - \tilde{m}_+}{(p-k)^2 - \tilde{m}_+{}^2}\right] \gamma^\mu \frac{-\Lambda^2}{k^2(k^2 - \Lambda^2)} d^4k \quad (2.8)$$

Making use of the formulas

$$\gamma_\mu \gamma^\mu = 4 \qquad (2.9)$$

$$\gamma_\mu \gamma_\nu \gamma^\mu = -2\gamma_\nu \qquad (2.10)$$

$$\frac{1}{ab} = \int_0^1 \frac{dx}{[ax + b(1-x)]^2} \qquad (2.11)$$

$$\frac{\Lambda^2}{k^2(k^2 - \Lambda^2)} = \int_{-\Lambda^2}^0 \frac{dL}{(k^2 + L)^2} \qquad (2.12)$$

we obtain

$$\Sigma = \frac{(\tilde{m}_+{}^2 + \tilde{m}_-{}^2)^{1/2}}{\tilde{m}_+ + \tilde{m}_-} (A_1 - A_2) \qquad (2.13)$$

where

$$A_1 = -\int_{-\Lambda^2}^0 dL \int d^4k \frac{2\tilde{m}_- + 2\not{k}}{(k^2 - 2pk)(k^2 + L)^2} \qquad (2.14)$$

$$A_2 = -\int_{-\Lambda^2}^0 dL \int d^4k \frac{-4\tilde{m}_+ - 2\tilde{m}_- + 2\not{k}}{(k^2 - 2pk + \tilde{m}_-{}^2 - \tilde{m}_+{}^2)(k^2 + L)^2} \qquad (2.15)$$

In the derivation of equations (2.14) and (2.15), we used the fact, $\not{p}u = \tilde{m}_-u$ for the free-electron state. The integral with respect to k can easily be carried out by making use of the formulas given in the Appendix:

$$A_1 = -\int_{-\Lambda^2}^{0} dL \int_0^1 dx \, \frac{i\pi^2 2\tilde{m}_- x(2-x)}{xL - (1-x)^2 p^2}$$

$$= \int_{-\Lambda^2}^{0} dL \int_0^1 dy \, \frac{2i\pi^2 \tilde{m}_-(1-y)(1+y)}{\tilde{m}_-^2 y^2 - (1-y)L}$$

$$= (-2i\pi^2 \tilde{m}_-) \int_0^1 dy (1+y) \ln \frac{y^2}{y^2 - (\Lambda^2/\tilde{m}_-^2) y + (\Lambda^2/\tilde{m}_-^2)} \tag{2.16}$$

Here, we transformed as $x = 1 - y$. The integral (2.16) is trivial and yields

$$A_1 = 2i\pi^2 \tilde{m}_- \bar{A}_1 \tag{2.17}$$

where

$$\bar{A}_1 = \tfrac{3}{2} \ln \frac{\Lambda^2}{\tilde{m}_-^2} + W(\theta_+) + W(\theta_-) \tag{2.18}$$

$$W(\theta) = -\frac{1}{2\theta} + \frac{(3\theta + 1)(\theta - 1)}{2\theta^2} \ln (1 - \theta) \tag{2.19}$$

$$\theta_\pm = \frac{1}{2}\left[1 \pm \left(1 - \frac{4\tilde{m}_-^2}{\Lambda^2}\right)^{1/2}\right] \tag{2.20}$$

This result is exactly the same one as obtained by Feynman (1949) with a cutoff.

Similarly, the integral (2.15) can easily be carried out and the result is

$$A_2 = -2i\pi^2 \int_0^1 dy(-2\tilde{m}_+ - \tilde{m}_- + \tilde{m}_- y)$$

$$\times \ln \frac{y(\tilde{m}_-^2 y - \tilde{m}_-^2 + \tilde{m}_+^2)}{\tilde{m}_-^2 y^2 - (\tilde{m}_-^2 - \tilde{m}_+^2 + \Lambda^2) y + \Lambda^2}$$

$$= -2i\pi^2 \bar{A}_2 \tilde{m}_- \tag{2.21}$$

where

$$\bar{A}_2 = \frac{1}{\rho}\Bigg\{(2 + \tfrac{3}{4}\rho) + (2+\rho)\left[1 + \frac{1-\rho^2}{\rho^2} \ln (1 - \rho^2)\right]$$

$$+ \frac{2 - 3\rho^2}{4\rho} + \frac{(1-\rho^2)^2}{2\rho^3} \ln (1 - \rho^2)$$

$$+ (2 + \tfrac{1}{2}\rho) \ln \rho^2 + (2 + \rho)\Gamma_1 - \rho\Gamma_2\Bigg\} \tag{2.22}$$

$$\rho = \tilde{m}_-/\tilde{m}_+ \tag{2.23}$$

$$\Gamma_1 = -\ln \rho^2 + \frac{\Lambda^2}{\tilde{m}_+^2 - \Lambda^2} \ln \frac{\tilde{m}_+^2}{\Lambda^2} - 1 \tag{2.24}$$

$$\Gamma_2 = -\tfrac{1}{2} \ln \rho^2 - \frac{\Lambda^4}{2(\tilde{m}_+ - \Lambda^2)^2} \ln \frac{\tilde{m}_+^2}{\Lambda^2} - \frac{1}{4} + \frac{\Lambda^2}{2(\tilde{m}_+^2 - \Lambda^2)} \tag{2.25}$$

Thus, the electron self-energy is given in a simple form

$$\Delta m = \frac{\alpha \tilde{m}_-}{2\pi} \frac{(\tilde{m}_+{}^2 + \tilde{m}_-{}^2)^{1/2}}{\tilde{m}_+ + \tilde{m}_-} (\bar{A}_2 + \bar{A}_1) \tag{2.26}$$

For the case of $\tilde{m}_+ \gg \tilde{m}_-$

$$\frac{(\tilde{m}_+{}^2 + \tilde{m}_-{}^2)^{1/2}}{\tilde{m}_+ + \tilde{m}_-} = 1 - \rho + \tfrac{3}{2}\rho^2 + \cdots \tag{2.27}$$

$$\bar{A}_1 = \tfrac{3}{2} \ln \frac{\Lambda^2}{\tilde{m}_-{}^2} + \tfrac{3}{4} - \tfrac{3}{4}\rho^2 - \cdots \tag{2.28}$$

$$\bar{A}_2 = \frac{1}{\rho}\left[(2 + \tfrac{3}{4}\rho + \rho^2 + \cdots) + \left(2 + \frac{\rho}{2}\right) \ln \rho^2 + (2 + \rho)\Gamma_1 - \rho\Gamma_2\right] \tag{2.29}$$

As was expected, the electron self-energy due to the electromagnetic interaction is finite. Therefore, it is possible to estimate the mass increase (decrease) of a particle when a charge is given. We propose here a model in which an electron is created by giving the charge to a neutrino and the mass difference comes from the self-energy due to the pure electromagnetic interaction. With $\tilde{m}_+ = 3.45 \times 10^6$ GeV, $\Lambda = 197$ GeV, $\tilde{m}_-$ (neutrino mass) $= 60$ eV, and $\alpha = 1/137.03608$, we obtain $\Delta m = 0.51$ MeV. Although we consider only the lowest-order term, the neutrino–electron mass difference could be obtained by adjusting the values of $\tilde{m}_+$ and Λ when the higher-order terms are taken into account. However, these quantities should be discussed together with the proton–neutron mass difference and the anomalous magnetic moments of the electron and the muon. Numerical results will be given in Section 5.

3. VACUUM POLARIZATION

In this section we shall calculate the vacuum polarization and show that this quantity is also finite in our theory.

The lowest order of vacuum polarization shown in Figure 2 is described by the following integrals:

$$\begin{aligned}\Pi_{\mu\nu}(k) = {} & \frac{i\alpha}{4\pi^3} \frac{\tilde{m}_+{}^2 + \tilde{m}_-{}^2}{(\tilde{m}_+ + \tilde{m}_-)^2} \\ & \times \left\{ \mathrm{Tr} \int \frac{\gamma_\mu(\not{p} + \not{k} + \tilde{m}_-)\gamma_\nu(\not{p} + \tilde{m}_-)}{[(p+k)^2 - \tilde{m}_-{}^2](p^2 - \tilde{m}_-{}^2)} d^4p \right. \\ & - \mathrm{Tr} \int \frac{\gamma_\mu(\not{p} + \not{k} - \tilde{m}_+)\gamma_\nu(\not{p} + \tilde{m}_-)}{[(p+k)^2 - \tilde{m}_+{}^2](p^2 - \tilde{m}_-{}^2)} d^4p \\ & - \mathrm{Tr} \int \frac{\gamma_\mu(\not{p} + \not{k} + \tilde{m}_-)\gamma_\nu(\not{p} - \tilde{m}_+)}{[(p+k)^2 - \tilde{m}_-{}^2](p^2 - \tilde{m}_+{}^2)} d^4p \\ & \left. + \mathrm{Tr} \int \frac{\gamma_\mu(\not{p} + \not{k} - \tilde{m}_+)\gamma_\nu(\not{p} - \tilde{m}_+)}{[(p+k)^2 - \tilde{m}_+{}^2](p^2 - \tilde{m}_+{}^2)} d^4p \right\}\end{aligned} \tag{3.1}$$

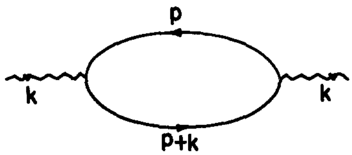


Fig. 2. The lowest-order Feynman graph for the vacuum polarization.

where Tr is to take the trace of matrices. Making use of the formula (2.11), we can rewrite the integrals (3.1) in the form

$$\Pi_{\mu\nu}(k) = \frac{i\alpha}{4\pi^3}\frac{\tilde{m}_+^2+\tilde{m}_-^2}{(\tilde{m}_++\tilde{m}_-)^2}\left\{\mathrm{Tr}\int_0^1 dz\int d^4p\,\frac{\gamma_\mu(\not{p}+\not{k}+\tilde{m}_-)\gamma_\nu(\not{p}+\tilde{m}_-)}{[(p+kz)^2+k^2(z-z^2)-\tilde{m}_-^2]^2}\right.$$
$$-\mathrm{Tr}\int_0^1 dz\int d^4p\,\frac{\gamma_\mu(\not{p}+\not{k}-\tilde{m}_+)\gamma_\nu(\not{p}+\tilde{m}_-)}{[(p+kz)^2+k^2(z-z^2)-(\tilde{m}_+^2-\tilde{m}_-^2)z-\tilde{m}_-^2]^2}$$
$$-\mathrm{Tr}\int_0^1 dz\int d^4p\,\frac{\gamma_\mu(\not{p}+\not{k}+\tilde{m}_-)\gamma_\nu(\not{p}-\tilde{m}_+)}{[(p+kz)^2+k^2(z-z^2)-(\tilde{m}_-^2-\tilde{m}_+^2)z-\tilde{m}_+^2]^2}$$
$$\left.+\mathrm{Tr}\int_0^1 dz\int d^4p\,\frac{\gamma_\mu(\not{p}+\not{k}-\tilde{m}_+)\gamma_\nu(\not{p}-\tilde{m}_+)}{[(p+kz)^2+k^2(z-z^2)-\tilde{m}_+^2]^2}\right\} \tag{3.2}$$

Shifting the origin of interaction from p to $p - kz$ considering the relations

$$\begin{aligned}\mathrm{Tr}\,(\gamma_\mu\not{p}\gamma_\nu\not{p}) &= -2g_{\mu\nu}p^2\\ \mathrm{Tr}\,(\gamma_\mu\gamma_\nu) &= 4g_{\mu\nu}\\ \mathrm{Tr}\,(\gamma_\mu k\gamma_\nu k) &= 8k_\mu k_\nu - 4g_{\mu\nu}k^2\\ \mathrm{Tr}\,(\gamma_\mu\gamma_\nu\gamma_\lambda) &= 0\end{aligned} \tag{3.3}$$

we obtain

$$\Pi_{\mu\nu}(k) = \frac{i\alpha}{4\pi^3}\frac{\tilde{m}_+^2+\tilde{m}_-^2}{(\tilde{m}_++\tilde{m}_-)^2}$$
$$\times\left\{\int_0^1 dz\int d^4p\,\frac{-(8k_\mu k_\nu-4g_{\mu\nu}k^2)(z-z^2)+g_{\mu\nu}(-2p^2+4\tilde{m}_-^2)}{[p^2+k^2(z-z^2)-\tilde{m}_-^2]^2}\right.$$
$$-\int_0^1 dz\int d^4p\,\frac{-(8k_\mu k_\nu-4g_{\mu\nu}k^2)(z-z^2)+g_{\mu\nu}(-2p^2-4\tilde{m}_+\tilde{m}_-)}{[p^2+k^2(z-z^2)-(\tilde{m}_+^2-\tilde{m}_-^2)z-\tilde{m}_-^2]^2}$$
$$-\int_0^1 dz\int d^4p\,\frac{-(8k_\mu k_\nu-4g_{\mu\nu}k^2)(z-z^2)+g_{\mu\nu}(-2p^2-4\tilde{m}_-\tilde{m}_+)}{[p^2+k^2(z-z^2)-(\tilde{m}_-^2-\tilde{m}_+^2)z-\tilde{m}_+^2]^2}$$
$$\left.+\int_0^1 dz\int d^4p\,\frac{-(8k_\mu k_\nu-4g_{\mu\nu}k^2)(z-z^2)+g_{\mu\nu}(-2p^2+4\tilde{m}_+^2)}{[p^2+k^2(z-z^2)-\tilde{m}_+^2]^2}\right\} \tag{3.4}$$

where we have taken account of the fact that the terms odd in p drop by integration. $\Pi_{\mu\nu}(0)$ gives the photon mass (i.e., the photon self-energy). Although it should be zero, the integrals (3.4) yield a divergent result for $k = 0$. In order to overcome this difficulty, let us impose a gauge invariance on it as is done in the usual theory, i.e., $k^\mu \Pi_{\mu\nu}(k) = \Pi_{\mu\nu}(k)k^\nu = 0$. Thus,

$$\begin{aligned}
0 &= k^\mu \Pi_{\mu\nu}(k) \\
&= \frac{i\alpha}{4\pi^3} \frac{\tilde{m}_+{}^2 + \tilde{m}_-{}^2}{(\tilde{m}_+ + \tilde{m}_-)^2} k_\nu \left\{ \int_0^1 dz \int d^4p \, \frac{-4k^2(z - z^2) + (-2p^2 + 4\tilde{m}_-{}^2)}{[p^2 + k^2(z - z^2) - \tilde{m}_-{}^2]^2} \right. \\
&\quad - \int_0^1 dz \int d^4p \, \frac{-4k^2(z - z^2) + (-2p^2 - 4\tilde{m}_+\tilde{m}_-)}{[p^2 + k^2(z - z^2) - (\tilde{m}_+{}^2 - \tilde{m}_-{}^2)z - m_-{}^2]^2} \\
&\quad - \int_0^1 dz \int d^4p \, \frac{-4k^2(z - z^2) + (-2p^2 - 4\tilde{m}_+\tilde{m}_-)}{[p^2 + k^2(z - z^2) - (\tilde{m}_-{}^2 - \tilde{m}_+{}^2)z - \tilde{m}_+{}^2]^2} \\
&\quad \left. + \int_0^1 dz \int d^4p \, \frac{-4k^2(z - z^2) + (-2p^2 + 4\tilde{m}_+{}^2)}{[p^2 + k^2(z - z^2) - \tilde{m}_+{}^2]^2} \right\}
\end{aligned} \tag{3.5}$$

On the other hand, since $\Pi_{\mu\nu}(k)$ is a tensor of second rank, we can write it down in the form

$$\Pi_{\mu\nu}(k) = k_\mu k_\nu C(k^2) + k^2 g_{\mu\nu} D(k^2) \tag{3.6}$$

Considering that $C(k^2) + D(k^2) = 0$ by the gauge invariance, we find

$$g^{\mu\nu}\Pi_{\mu\nu}(k) = k^2 C(k^2) + 4k^2 D(k^2) = -3k^2 C(k^2) \tag{3.7}$$

Thus, the vacuum polarization is expressed as

$$\Pi_{\mu\nu}(k) = (k_\mu k_\nu - k^2 g_{\mu\nu}) C(k^2) \tag{3.8}$$

Calculating the quantity $g^{\mu\nu}\Pi_{\mu\nu}(k)$ with (3.4) and taking (3.5) into account, we obtain the expression of $C(k^2)$ as

$$\begin{aligned}
C(k^2) &= -\frac{2i\alpha}{\pi^3} \frac{\tilde{m}_+{}^2 + \tilde{m}_-{}^2}{(\tilde{m}_+ + \tilde{m}_-)^2} \left\{ \int_0^1 dz \int d^4p \, \frac{z - z^2}{[p^2 + k^2(z - z^2) - \tilde{m}_-{}^2]^2} \right. \\
&\quad - \int_0^1 dz \int d^4p \, \frac{z - z^2}{[p^2 + k^2(z - z^2) - (\tilde{m}_+{}^2 - \tilde{m}_-{}^2)z - \tilde{m}_-{}^2]^2} \\
&\quad - \int_0^1 dz \int d^4p \, \frac{z - z^2}{[p^2 + k^2(z - z^2) - (\tilde{m}_-{}^2 - \tilde{m}_+{}^2)z - \tilde{m}_+{}^2]^2} \\
&\quad \left. + \int_0^1 dz \int d^4p \, \frac{z - z^2}{[p^2 + k^2(z - z^2) - \tilde{m}_+{}^2]^2} \right\}
\end{aligned} \tag{3.9}$$

The form of $\Pi_{\mu\nu}(k)$ obtained in (3.8) and (3.9) is gauge invariant and has a guarantee that the photon mass is zero, as long as $C(k^2)$ is finite. Using the methods given in the paper by Feynman (1949), one can evaluate the integrals (3.9). The result is

$$C(k^2) = -\frac{2i\alpha}{\pi^3}\frac{\tilde{m}_+{}^2 + \tilde{m}_-{}^2}{(\tilde{m}_+ + \tilde{m}_-)^2}[A(\tilde{m}_+, \tilde{m}_-; k) + A(\tilde{m}_-, \tilde{m}_+; k)] \tag{3.10}$$

where

$$A(\tilde{m}_\pm, \tilde{m}_\mp; k) = 4i\pi^2\Delta\int_0^1 dz\int_0^1 dy\,\frac{(z^2 - z^3)(y - y^2)}{\Delta zy - k^2(z - z^2) + \tilde{m}_\mp{}^2}$$
$$+ i\pi^2\Delta^2\int_0^1 dz\int_0^1 dy\,\frac{(z^3 - z^4)(y - y^2)(1 - 2y)}{[\Delta zy - k^2(z - z^2) + \tilde{m}_\mp{}^2]^2} \tag{3.11}$$

with

$$\Delta = \tilde{m}_+{}^2 - \tilde{m}_-{}^2 \tag{3.12}$$

After integrating with respect to y, A becomes

$$A(\tilde{m}_+, \tilde{m}_-; k) = i\pi^2\left[\frac{1}{2} + \frac{\tilde{m}_-{}^2}{\Delta}\left(1 - \frac{k^2}{6\tilde{m}_-{}^2}\right) - \frac{2\tilde{m}_-{}^4}{\Delta^2}B_{-1}\right.$$
$$+ \frac{4\tilde{m}_-{}^4 - 2\tilde{m}_+{}^2\tilde{m}_-{}^2 + 4\tilde{m}_-{}^2k^2}{\Delta^2}B_0$$
$$- \frac{\tilde{m}_-{}^4 - \tilde{m}_+{}^4 + k^2(10\tilde{m}_-{}^2 - 2\tilde{m}_+{}^2) + 2k^4}{\Delta^2}B_1$$
$$+ \frac{-\tilde{m}_-{}^4 + 2\tilde{m}_+{}^2\tilde{m}_-{}^2 - \tilde{m}_+{}^4 + k^2(8\tilde{m}_-{}^2 - 4\tilde{m}_+{}^2) + 6k^4}{\Delta^2}B_2$$
$$\left. - \frac{(2\tilde{m}_-{}^2 - 2\tilde{m}_+{}^2)k^2 + 6k^4}{\Delta^2}B_3 + \frac{2k^4}{\Delta^2}B_4\right] \tag{3.13}$$

where

$$B_j = \int_0^1 z^j \ln\left[\frac{\Delta z - k^2(z - z^2) + \tilde{m}_-{}^2}{\tilde{m}_-{}^2 - k^2(z - z^2)}\right]dz \tag{3.14}$$

For the case of $\tilde{m}_+ \gg \tilde{m}_-$ and $\tilde{m}_-{}^2 \gg k^2$ we obtain

$$(A\tilde{m}_+, \tilde{m}_-; k) = i\pi^2\left\{-(\rho^4 + 2\rho^6 + \cdots)\left[\left(\ln\frac{1}{\rho^2}\right)^2 + \frac{\pi^2}{6}\right]\right.$$
$$+ \frac{13}{36} + \frac{17}{6}\rho^2 + \frac{13}{6}\rho^4 + \cdots$$
$$+ \left(\frac{1}{6} - \rho^2 - \frac{3}{2}\rho^4 - \cdots\right)\ln\frac{1}{\rho^2}$$

$$+ \frac{k^4}{\tilde{m}_-{}^2}\left[\left(\frac{1}{6}\rho^2 + \frac{1}{2}\rho^4 + \cdots\right)\ln\frac{1}{\rho^2} + \frac{1}{30} - \frac{197}{360}\rho^2 + \cdots\right]$$

$$+ \frac{k^4}{\tilde{m}_-{}^4}\left[\left(\frac{29}{10}\rho^4 + \cdots\right)\ln\frac{1}{\rho^2} + \frac{1}{280} + \frac{1}{60}\rho^2 + \cdots\right]$$

$$+ O\left(\frac{k^6}{\tilde{m}_-{}^6}\right)\Bigg\} \tag{3.15}$$

$$A(\tilde{m}_-, \tilde{m}_+; k) = i\pi^2\Bigg\{\frac{\pi^2}{3} - \frac{113}{36} + \left(\frac{2\pi^2}{3} - \frac{37}{6}\right)\rho^2 + \left(\pi^2 - \frac{64}{3}\right)\rho^4 + \cdots$$

$$+ \left(4\rho^2 + \frac{37}{2}\rho^4 + \cdots\right)\ln\frac{1}{\rho^2}$$

$$+ \frac{k^2}{\tilde{m}_-{}^2}\Bigg[(33\rho^4 + 81\rho^6 + \cdots) \times \ln\frac{1}{\rho^2} + \frac{2587}{360}\rho^2 - \frac{1}{72}\rho^4 - \cdots\Bigg]$$

$$+ \frac{k^4}{\tilde{m}_-{}^4}\Bigg[-\left(\frac{4}{5}\rho^4 + \frac{48}{5}\rho^6 + \cdots\right) \times \ln\frac{1}{\rho^2} - \frac{17471}{4200}\rho^4 - \cdots\Bigg] + O\left(\frac{k^6}{m_-{}^6}\right)\Bigg\} \tag{3.16}$$

where $\rho = \tilde{m}_-/\tilde{m}_+$.

Thus, considering that

$$\frac{\tilde{m}_+{}^2 + \tilde{m}_-{}^2}{(\tilde{m}_+ + \tilde{m}_-)^2} = 1 - 2\rho + 4\rho^2 + \cdots \tag{3.17}$$

we find the final expression for $C(k^2)$ as

$$C(k^2) = \frac{\alpha}{\pi}\Bigg\{-\left(\frac{1}{3} + \frac{2}{3}\rho + \frac{22}{3}\rho^2 + \cdots\right)\ln\rho^2$$

$$+ \left(\frac{2\pi^2}{3} - \frac{50}{9}\right) - \left(\frac{4\pi^2}{3} - \frac{100}{9}\right)\rho + \left(4\pi^2 - \frac{260}{9}\right)\rho^2 + \cdots$$

$$+ \frac{k^2}{\tilde{m}_-{}^2}\left[\left(\frac{2}{3}\rho^2 + \cdots\right)\ln\rho^2 + \frac{1}{15} - \frac{2}{15}\rho + \frac{1219}{90}\rho^2 + \cdots\right]$$

$$+ \frac{k^4}{\tilde{m}_-{}^4}\left[-\left(\frac{21}{5}\rho^4 + \cdots\right)\ln\rho^2 + \frac{1}{140} - \frac{1}{70}\rho - \frac{13}{210}\rho^2 - \cdots\right]$$

$$+ O\left(\frac{k^6}{\tilde{m}_-{}^6}\right)\Bigg\} \tag{3.18}$$

This result is obviously finite. Therefore, the photon mass vanishes in the framework of our present theory. The observed external charge-current density is given in terms of the bare external charge-current density $J_\mu{}^e(x)$ as

$$J^e_{\mu R}(x) = \left\{1 + \frac{\alpha}{\pi}\left[-\left(\frac{1}{3} + \frac{2}{3}\rho + \frac{22}{3}\rho^2 + \cdots\right)\ln\rho^2 + \left(\frac{2\pi^2}{3} - \frac{50}{9}\right) - \left(\frac{4\pi^2}{3} - \frac{100}{9}\right)\rho + \left(4\pi^2 - \frac{260}{9}\right)\rho^2 + \cdots\right]\right\}J_\mu{}^e(x) \tag{3.19}$$

with $\tilde{m}_- = 0.511$ MeV, $\tilde{m}_+ = 3.45 \times 10^6$ GeV, and $\alpha = 1/137.03608$, we find that the correction due to vacuum polarization is 3.7%, i.e.,

$$J^e_{\mu R}(x) = 1.037 \times J_\mu{}^2(x)$$

The effective potential seen by the electron as a result of vacuum polarization effects is described as follows:

$$a^{\text{eff}}_\mu(k) = \left\{1 + \frac{\alpha k^2}{15\pi\tilde{m}_-{}^2}\left[(10\rho^2 + \cdots)\ln\rho^2 + 1 - 2\rho + \frac{1219}{6}\rho^2 + \cdots\right]\right\}a_\mu(k) \tag{3.20}$$

and in configuration space

$$A^{\text{eff}}_\mu(x) = \left\{1 - \frac{\alpha}{15\pi\tilde{m}_-{}^2}\left[(10\rho^2 + \cdots)\ln\rho^2 + 1 - 2\rho + \frac{1219}{6}\rho^2 + \cdots\right]\Box\right\}A_\mu(x) \tag{3.21}$$

where $\Box = \partial_\mu\partial^\mu$ is the D'Alembertian operator. When all terms depending on ρ are neglected, our results reduce to those obtained in the usual theory.

4. VERTEX CORRECTION

In this section, we shall calculate the lowest-order Feynman graph shown in Figure 3. This correction is particularly important because the anomalous magnetic moment is involved. Tremendously precise values for $(g - 2)$ of

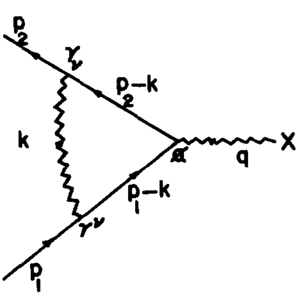


Fig. 3. Feynman graph for vertex correction.

electron and muon are now available in both experiment and theory (Combley and Picasso, 1974). The new data (Bailey et al., 1975) reveal a discrepancy between the experimental and theoretical values for the muon anomalous magnetic moment. This discrepancy might be an evidence for a breakdown of the usual QED. Thus it is extremely interesting to see the results induced from our theory.

The vertex correction is obtained by calculation of the following matrix element:

$$\Lambda(p_2, p_1) = -\frac{i\alpha}{4\pi^3}\frac{\tilde{m}_+{}^2 + \tilde{m}_-{}^2}{(\tilde{m}_+ + \tilde{m}_-)^2}\int \gamma_\nu\left(\frac{1}{\not{p}_2 - \not{k} - \tilde{m}_-} - \frac{1}{\not{p}_2 - \not{k} + \tilde{m}_+}\right)\not{a}$$
$$\times\left(\frac{1}{\not{p}_1 - \not{k} - \tilde{m}_-} - \frac{1}{\not{p}_1 - \not{k} + \tilde{m}_+}\right)\gamma^\nu\left(\frac{1}{k^2} - \frac{1}{k^2 - \Lambda^2}\right)d^4k \tag{4.1}$$

By the formulas (2.11) and (2.12), equation (4.1) can be rewritten as

$$\Lambda(p_2, p_1) = \frac{i\alpha}{4\pi^3}\frac{\tilde{m}_+{}^2 + \tilde{m}_-{}^2}{(\tilde{m}_+ + \tilde{m}_-)^2}$$
$$\times\,[G(\tilde{m}_-, \tilde{m}_-) - G(\tilde{m}_+, \tilde{m}_-) - G(\tilde{m}_-, \tilde{m}_+) + G(\tilde{m}_+, \tilde{m}_+)] \tag{4.2}$$

where

$$G(\xi, \eta) = \int_{\lambda_m{}^2}^{\Lambda^2} dL \int d^4k \int_0^1 dy\, \frac{\gamma_\nu(\not{p}_2 - \not{k} + \xi')\not{a}(\not{p}_1 - \not{k} + \eta')\gamma^\nu}{(k^2 - 2kp_y - \Delta_y)^2(k^2 - L)^2} \tag{4.3}$$

with

$$\Delta_y = (\xi^2 - \tilde{m}_-{}^2) + (\eta^2 - \xi^2)y \tag{4.4}$$

$$p_y = p_2 + (p_1 - p_2)y = p_2 - qy \tag{4.5}$$

$$\xi', \eta' = \mp\tilde{m}_\pm \quad \text{for} \quad \xi, \eta = \tilde{m}_\pm \tag{4.6}$$

The lower limit $\lambda_m{}^2$ in the integral (4.3) is temporarily introduced to avoid divergences due to the well-known infrared catastrophy (Feynman, 1949). This catastrophy can actually be resolved by summing up all graphs (Feynman, 1961). Making use of the following formula together with (2.9) and (2.10),

$$\gamma_\mu\not{a}\not{b}\gamma^\mu = 2(\not{a}\not{b} + \not{b}\not{a}) \tag{4.7}$$

$$\gamma_\mu\not{a}\not{b}\not{d}\gamma^\mu = -2\not{d}\not{b}\not{a} \tag{4.8}$$

we can easily find

$$\gamma_\nu(\not{p}_2 - \not{k} + \xi')\not{a}(\not{p}_1 - \not{k} + \eta')\gamma^\nu$$
$$= -2\not{p}_1\not{a}\not{p}_2 + 2\eta'(\not{p}_2\not{a} + \not{a}\not{p}_2) + 2\xi'(\not{a}\not{p}_1 + \not{p}_1\not{a}) - 2\xi'\eta'\not{a}$$
$$+ 2\not{k}\not{a}\not{p}_2 + 2\not{p}_1\not{a}\not{k} - 2(\xi' + \eta')(\not{a}\not{k} + \not{k}\not{a}) - 2\not{k}\not{a}\not{k} \tag{4.9}$$

Integration with respect to k can be carried out with the help of the formulas (A.10) and (A.11) in the Appendix:

$$G(\xi,\eta) = i\pi^2\int_{\lambda_m^2}^{\Lambda^2} dL \int_0^1 dy \int_0^1 dx \frac{x(1-x)g(\xi',\eta')}{[x\Delta_y + (1-x)L + x^2p_y{}^2]^2} \tag{4.10}$$

$$\begin{aligned} g(\xi',\eta') = {} & -2\not{p}_1\not{a}\not{p}_2 + 2\eta'(\not{p}_2\not{a} + \not{a}\not{p}_2) + 2\xi'(\not{p}_1\not{a} + \not{a}\not{p}_1) - 2\xi'\eta'\not{a} \\ & + x[2\not{p}_y\not{a}\not{p}_2 + 2\not{p}_1\not{a}\not{p}_y - 2(\xi' + \eta')(\not{p}_y\not{a} + \not{a}\not{p}_y)] \\ & - 2\{x^2\not{p}_y\not{a}\not{p}_y + \not{a}[x\Delta_y + (1-x)L + x^2p_y{}^2]\} \end{aligned} \tag{4.11}$$

Considering $(p_{1,2}^2 - \tilde{m}_-{}^2)u = 0$ for a free particle state u and with the formulas (2.9), (2.10), (4.7), and (4.8), we rewrite (4.11) in the form

$$\begin{aligned} g(\xi',\eta') = {} & -2[\xi'\eta' + q^2 + \tilde{m}_-{}^2 + 2\tilde{m}_-(\tilde{m}_- - \xi' - \eta')]\not{a} \\ & - 2\tilde{m}_-(\not{a}\not{q} - \not{q}\not{a}) + 2(\eta'\not{a}\not{q} - \xi'\not{q}\not{a}) \\ & + 2x\{[q^2 + 2\tilde{m}_-(\tilde{m}_- - \xi' - \eta')]\not{a} \\ & \qquad + [\tilde{m}_- + (\tilde{m}_- - \xi' - \eta')(1-y)]\not{a}\not{q} \\ & \qquad - [\tilde{m}_- + (\tilde{m}_- - \xi' - \eta')y]\not{q}\not{a}\} \\ & - 2x^2\{[\tilde{m}_-{}^2 + q^2(y - y^2)]\not{a} + \tilde{m}_-(1-y)\not{a}\not{q} - \tilde{m}_-y\not{q}\not{a}\} \\ & - 2\not{a}[x\Delta_y + (1-x)L + x^2p_y{}^2] \end{aligned} \tag{4.12}$$

The integral (4.10) with respect to L is first carried out as follows:

$$\int_{\lambda_m^2}^{\Lambda^2} \frac{y^m x^n(1-x)}{[x\Delta_y + (1-x)L + x^2p_y{}^2]^2}\,dL = \frac{y^m x^n}{x\Delta_y + (1-x)\lambda_m{}^2 + x^2p_y{}^2} - \frac{y^m x^n}{x\Delta_y + (1-x)\Lambda^2 + x^2p_y{}^2} \tag{4.13}$$

$$\int_{\lambda_m^2}^{\Lambda^2} \frac{x(1-x)}{x\Delta_y + (1-x)L + x^2p_y{}^2}\,dL = x\ln\frac{x\Delta_y + x^2p_y{}^2 + (1-x)\Lambda^2}{x\Delta_y + x^2p_y{}^2 + (1-x)\lambda_m{}^2} \tag{4.14}$$

and the expansion in terms of q^2,

$$\frac{1}{x\Delta_y + (1-x)\lambda_m{}^2 + x^2p_y{}^2} = \frac{1}{x\Delta_y + (1-x)\lambda_m{}^2 + \tilde{m}_-{}^2x^2} + \frac{q^2x^2(y-y^2)}{[x\Delta_y + (1-x)\lambda_m{}^2 + \tilde{m}_-{}^2x^2]^2} + \cdots \tag{4.15}$$

is made use of to reach a final expression of $\Lambda(p_2, p_1)$. The other integrals are straightforward although tedious. The result is

$$\begin{aligned} & [G(\tilde{m}_-, \tilde{m}_-) - G(\tilde{m}_+, \tilde{m}_-) - G(\tilde{m}_-, \tilde{m}_+) + G(\tilde{m}_+, \tilde{m}_+)] \\ & \quad = i\pi^2\left[\not{a}F_0(\tilde{m}_+, \tilde{m}_-, \Lambda) - \not{a}\frac{4q^2}{3\tilde{m}_-{}^2}F_1(\tilde{m}_+, \tilde{m}_-, \Lambda) + \frac{\not{a}\not{q} - \not{q}\not{a}}{2\tilde{m}_-}F_2(\tilde{m}_+, \tilde{m}_-, \Lambda)\right] \end{aligned} \tag{4.16}$$

where

$$
\begin{aligned}
F_0(\tilde{m}_+, \tilde{m}_-, \Lambda) = \Bigg[&-(1 + 9\epsilon^4 + 48\epsilon^6 + \cdots) \\
&\times \ln \frac{1}{\epsilon^2} - \left(\frac{9}{2} + 4\epsilon^2 + \frac{69}{4}\epsilon^4 + 368\epsilon^6 + \cdots\right) + 2 \ln \frac{\tilde{m}_-^2}{\lambda_m^2} \Bigg] \\
&+ \Bigg[\frac{(4\tilde{m}_+\tilde{m}_- - 4\tilde{m}_-^2)}{\Delta} (K_0 - J_0) - \frac{8\tilde{m}_+\tilde{m}_-}{\Delta} (K_1 - J_1) \\
&\quad + \frac{4\tilde{m}_-^2}{\Delta} (K_2 - J_2) + 4A_7^0 \Bigg] \\
&- [(2\tilde{m}_+^2 + 8\tilde{m}_+\tilde{m}_- + 2\tilde{m}_-^2)(R_{11} - R_{12}) \\
&\quad - (8\tilde{m}_+\tilde{m}_- + 4\tilde{m}_-^2)(R_{21} - R_{22}) \\
&\quad + 2\tilde{m}_-^2(R_{31} - R_{32}) + 2R_{00}]
\end{aligned}
\tag{4.17}
$$

$$
\begin{aligned}
F_1(\tilde{m}_+, \tilde{m}_-, \Lambda) = \Bigg[&\frac{1}{2} \ln \frac{\tilde{m}_-^2}{\lambda_m^2} - \left(\frac{3}{8} + \frac{7}{24}\epsilon^2 + \frac{43}{48}\epsilon^4 + \frac{301}{80}\epsilon^6 + \cdots\right) \\
&- \left(\frac{1}{4}\epsilon^2 - \frac{1}{4}\epsilon^4 - \frac{9}{4}\epsilon^6 - \cdots\right) \ln \frac{1}{\epsilon^2} \Bigg] \\
&- \Bigg[\frac{3\tilde{m}_-^2}{\Delta} (K_0 - J_0) - \frac{3\tilde{m}_-^2}{\Delta} (K_1 - J_1) + 3\tilde{m}_-^2 A_5^0 \\
&\quad - 3\tilde{m}_-^2 A_6^0 + 3\tilde{m}_-^2(R_{71} - R_{72}) \\
&\quad + (3\tilde{m}_+\tilde{m}_-^3 - 3\tilde{m}_-^4)(Q_{11} - Q_{12}) \\
&\quad - 6\tilde{m}_+\tilde{m}_-^3(Q_{21} - Q_{22}) + 3\tilde{m}_-^4(Q_{31} - Q_{32}) \Bigg] \\
&+ \Bigg[\frac{3\tilde{m}_-^2}{2} (R_{11} - R_{12}) + \left(\frac{\tilde{m}_+^2\tilde{m}_-^2}{4} + \tilde{m}_+\tilde{m}_-^3 + \frac{\tilde{m}_-^4}{4}\right) \\
&\quad \times (T_{11} - T_{12}) - \frac{3\tilde{m}_-^2}{2} (R_{21} - R_{22}) \\
&\quad + \frac{\tilde{m}_-^2}{4} (R_{31} - R_{32}) - \left(\tilde{m}_+\tilde{m}_-^3 + \frac{\tilde{m}_-^4}{2}\right) \\
&\quad \times (T_{21} - T_{22}) + \frac{\tilde{m}_-^4}{4} (T_{31} - T_{32}) \Bigg]
\end{aligned}
\tag{4.18}
$$

$$
\begin{aligned}
F_2(\tilde{m}_+, \tilde{m}_-, \Lambda) = \Bigg[&\left(1 - \frac{2}{3}\epsilon^2 - \frac{25}{6}\epsilon^4 - \frac{97}{5}\epsilon^6 - \cdots\right) \\
&+ (2\epsilon^4 + 12\epsilon^6 + \cdots) \ln \frac{1}{\epsilon^2} \Bigg]
\end{aligned}
$$

$$+ \left[\frac{(4\tilde{m}_+\tilde{m}_- + 4\tilde{m}_-^2)}{\Delta}(K_0 - J_0) - \frac{(4\tilde{m}_+\tilde{m}_- + 8\tilde{m}_-^2)}{\Delta}(K_1 - J_1) + \frac{4\tilde{m}_-^2}{\Delta}(K_2 - J_2)\right]$$

$$- [(4\tilde{m}_+\tilde{m}_- + 4\tilde{m}_-^2)(R_{11} - R_{12}) - (4\tilde{m}_+\tilde{m}_- + 6\tilde{m}_-^2) \times (R_{21} - R_{22}) + 2\tilde{m}_-^2(R_{31} - R_{32})] \quad (4.19)$$

Notations used in (4.17)–(4.19) are given as

$$\epsilon = \tilde{m}_-/\Lambda \quad (4.20)$$

$$\rho = \tilde{m}_-/\tilde{m}_+ \quad (2.23)$$

$$\Delta = \tilde{m}_+^2 - \tilde{m}_-^2 \quad (3.12)$$

$$R_{n1} = \frac{1}{\tilde{m}_+^2}\left[\frac{1}{n} + \frac{\rho^2}{n(n+1)} + \frac{2\rho^6}{n(n+1)(n+2)} + \cdots\right] \quad (4.21)$$

$$R_{n2} = \int_0^1 \frac{x^n\,dx}{\tilde{m}_-^2x^2 + (\Delta - \Lambda^2)x + \Lambda^2}$$

$$T_{n1} = \frac{1}{\tilde{m}_+^2\tilde{m}_-^2}\left[\frac{\rho^2}{n+1} + \frac{2\rho^4}{(n+1)(n+2)} + \frac{6\rho^6}{(n+1)(n+2)(n+3)} + \cdots\right] \quad (4.23)$$

$$T_{n2} = \int_0^1 \frac{x^{n+2}\,dx}{[\tilde{m}_-^2x^2 + (\Delta - \Lambda^2)x + \Lambda^2]^2} \quad (4.24)$$

$$R_{00} = \int_0^1 x \ln\left[\frac{\tilde{m}_-^2x^2 + (\Delta - \Lambda^2)x + \Lambda^2}{\tilde{m}_-^2x^2 + \Delta x + \Lambda^2}\right] dx \quad (4.25)$$

$$A_5^0 = -\frac{\tilde{m}_-^2}{\Delta^2}K_3 + \frac{\Lambda^2}{\Delta^2}J_1 - \frac{\Lambda^2}{\Delta^2}J_2 + \frac{\tilde{m}_-^2}{\Delta^2}J_3 \quad (4.26)$$

$$A_6^0 = \frac{\Lambda^2}{6\Delta^2} + \frac{\tilde{m}_-^4}{\Delta^3}K_4 - \frac{\Lambda^4}{\Delta^3}J_0 + \frac{2\Lambda^4}{\Delta^3}J_1 - \frac{2\Lambda^2\tilde{m}_-^2 + \Lambda^4}{\Delta^3}J_2 + \frac{2\Lambda^2\tilde{m}_-^2}{\Delta^3}J_3 - \frac{\tilde{m}_-^4}{\Delta^3}J_4 \quad (4.27)$$

$$A_7^0 = -\left(\frac{3}{4} + \frac{\rho^2}{18} + \frac{\rho^4}{72} + \frac{\rho^6}{180} + \cdots\right) + \left(\frac{1}{2} + \frac{\rho^2}{3} + \frac{\rho^4}{3} + \frac{\rho^6}{3} + \cdots\right)\ln\frac{1}{\rho^2} - \frac{\tilde{m}_-^2}{\Delta}J_2 - \frac{\Delta - \Lambda^2}{\Delta}J_1 - \frac{\Lambda^2}{\Delta}J_0 + \frac{1}{2} - \int_0^1 x\ln\left(x^2 - \frac{1}{\epsilon^2}x + \frac{1}{\epsilon^2}\right)dx \quad (4.28)$$

$$K_{n-1} = \frac{1}{n}\ln\frac{1}{\rho^2} + \frac{1}{n^2} - \frac{\rho^2}{n(n+1)} - \frac{\rho^4}{n(n+1)(n+2)} - \frac{2\rho^6}{n(n+1)(n+2)(n+3)} - \cdots \tag{4.29}$$

$$J_{n-1} = \int_0^1 x^{n-1}\ln\left[\frac{\tilde{m}_-^{\,2}x^2 + (\tilde{m}_+^{\,2} - \tilde{m}_-^{\,2} - \Lambda^2)x + \Lambda^2}{\tilde{m}_-^{\,2}x^2 - \Lambda^2 x + \Lambda^2}\right]dx \tag{4.30}$$

$$Q_{11} = -\frac{4}{3\tilde{m}_+^{\,2}\tilde{m}_-^{\,2}}\left[\left(\frac{1}{8} + \frac{9}{8}\rho^2 + \frac{23}{8}\rho^4 + \frac{43}{8}\rho^6 + \cdots\right) - \left(\frac{3}{8}\rho^2 + \frac{5}{4}\rho^4 + \frac{21}{8}\rho^6 + \cdots\right)\ln\frac{1}{\rho^2}\right] \tag{4.31}$$

$$Q_{21} = \frac{4}{3\tilde{m}_+^{\,2}\tilde{m}_-^{\,2}}\left[\left(\frac{\rho^2}{3} + \frac{121}{96}\rho^4 + \frac{411}{160}\rho^6 + \cdots\right) + \left(\frac{4}{9}\rho^2 + \frac{14}{9}\rho^4 + \frac{10}{3}\rho^6 + \cdots\right)\ln\frac{1}{\rho^2}\right] \tag{4.32}$$

$$Q_{31} = -\frac{4}{3\tilde{m}_+^{\,2}\tilde{m}_-^{\,2}}\left[\left(\frac{9}{64}\rho^2 + \frac{507}{800}\rho^4 + \frac{2397}{1600}\rho^6 + \cdots\right) - \left(\frac{3}{16}\rho^2 + \frac{27}{40}\rho^4 + \frac{117}{80}\rho^6 + \cdots\right)\ln\frac{1}{\rho^2}\right] \tag{4.33}$$

$$Q_{n2} = \frac{4}{3\tilde{m}_+^{\,2}\tilde{m}_-^{\,2}}\left[\frac{3\Lambda^2\tilde{m}_+^{\,2}\tilde{m}_-^{\,2}}{4(n+1)\Delta^3} + \frac{3\Lambda^2\tilde{m}_+^{\,2}\tilde{m}_-^{\,2}}{2\Delta^3}J_{n-1} + \frac{3(\Delta - 2\Lambda^2)}{4\Delta^3}\tilde{m}_+^{\,2}\tilde{m}_-^{\,2}J_n + \frac{3\tilde{m}_+^{\,2}\tilde{m}_-^{\,4}}{2\Delta^3}J_{n+1}\right] \tag{4.34}$$

$$\begin{aligned}(R_{71} - R_{72}) = \frac{4}{3\tilde{m}_-^{\,2}}\Bigg[&\frac{21}{64}\rho^4 + \frac{1131}{800}\rho^6 + \cdots - \left(\frac{3}{16}\rho^4 + \frac{21}{40}\rho^6 + \cdots\right)\\ &\times \ln\frac{1}{\rho^2} - \frac{\Lambda^2\tilde{m}_-^{\,2}}{8\Delta^2} + \frac{3\Lambda^4\tilde{m}_-^{\,2}}{4\Delta^3}J_0\\ &- \frac{6\Lambda^4\tilde{m}_-^{\,2} - 3\Lambda^2\tilde{m}_-^{\,2}\Delta}{4\Delta^3}J_1\\ &- \frac{3\Lambda^2\tilde{m}_-^{\,2}\Delta - 6\tilde{m}_-^{\,4}\Lambda^2 - 3\Lambda^4\tilde{m}_-^{\,2}}{4\Delta^3}J_2\\ &+ \frac{3\Delta\tilde{m}_-^{\,4} - 6\Lambda^2\tilde{m}_-^{\,4}}{4\Delta^3}J_3 - \frac{3\tilde{m}_-^{\,6}}{4\Delta^3}J_4\Bigg]\end{aligned} \tag{4.35}$$

The quantity

$$\frac{\alpha}{2\pi}\frac{\tilde{m}_+{}^2 + \tilde{m}_-{}^2}{(\tilde{m}_+ + \tilde{m}_-)^2} F_2(\tilde{m}_+, \tilde{m}_-, \Lambda)$$

gives the anomalous magnetic moment. In the limit $\Lambda \to \infty$ and $\tilde{m}_+ \to \infty$ it reduces to the Schwinger term obtained by the usual theory.

It should be noticed that even in the absence of infinities, the physical quantities such as the mass and charge have to be renormalized. If the theory were in the plight where divergence problems would keep us from extracting a clear physical picture, renormalization might be questionable. If the theory were, however, divergence free, renormalization could completely be performed. The origin of renormalization is due to the fact that the state of the system is described in terms of unperturbed bare wave functions, whereas interaction between fields can never be switched off in the actual world. In addition, since only the bare mass (charge) plus the corrections to it can ever be observed, the observables must always be expressed in terms of the renormalized constants. Therefore it is reasonable in numerical calculation of the anomalous magnetic moment to use the experimental values for $\tilde{m}_-$ and α in the above formulas. Numerical results and discussions are given in the following section.

5. NUMERICAL RESULTS AND DISCUSSIONS

In this section we shall give our numerical results and discussions on the self-energy and the anomalous magnetic moments of the electron and the muon.

5.1. The Electron Self-Energy and the Mass Differences of Neutrino–Electron and Neutron–Proton. Mass difference between the proton and neutron is a long-standing problem. Feynman and Speisman (1954) are the first persons who tried to calculate the proton–neutron mass difference field-theoretically. They pointed out that it was possible to obtain a negative sign for $m_p - m_n$ by a suitable choice of cutoff parameters since the proton has the anomalous magnetic moment under effects of the strong interaction. Cini, Ferrari, and Gatto (1959) estimated this quantity as $m_p - m_n = +0.66$ MeV (experimental value is -1.2933 ± 0.0001 MeV) using the nucleon form factor determined by the *e-p* scattering. In spite of numerous calculations, even a negative sign has not been obtained, apart from its absolute value.

Our present theory gives a positive sign to the proton self-energy and its absolute value is quite large compared with the experimental data. However, when effects of the strong interaction are taken into account, the

TABLE 1. Electron self-energy, photon–neutron mass difference, and anomalous magnetic moments of the electron and the muon. The recommended value of α (Taylor et al., 1969), 1/37.03608 was used. The values given with notations in the first row are experimental values.

l_{ph}[a] (cm)	$l_{e\mu N}$[b] (cm)	$\tilde{m}_- \equiv m_{\nu_e}$ (eV) $\lesssim 60$	$\Delta m(e^{\pm})$ (MeV) 0.510946	Δm_{pn} (MeV) ($m_n = 939.550$) -1.294	$\Delta a(e^-)$ $(5.0 \pm 3.5) \times 10^{-9}$[c]	$\Delta a(\mu^+)$ $(-13 \pm 29) \times 10^{-9}$[d]
10^{-16}	0.5705×10^{-20}	50	0.510968	18.83	0.93×10^{-11}	-3.7×10^{-9}
		10^{-3}	0.510961			
		1	0.510961			
		10^3	0.511031			
		10^6	0.554317			
2×10^{-16}	0.2479×10^{-19}	50	0.511003	16.56	0.38×10^{-10}	-14.8×10^{-9}
0.5×10^{-16}	0.1322×10^{-20}	50	0.510869	21.10	0.23×10^{-11}	-0.92×10^{-9}

[a] $l_{ph} = \hbar/\Lambda c$.
[b] $l_{e\mu N} = \hbar/c\,(\tilde{m}_+^2 + \tilde{m}_-^2)^{1/2}$.
[c] Granger and Ford, 1972.
[d] Bailey et al., 1975.

absolute value will reasonably be reduced because the cross term between the Coulomb repulsive part and magnetic moment part has an opposite sign. A correct sign might be reproduced.

Let us consider a model in which the neutrino and the electron are the same family. When a charge is given to the neutrino without worrying about the mechanism of load, the charge creates its own electromagnetic field and the neutrino increases in mass through interaction with its own field. This mass increase is just the electron self-energy which was calculated in Section 2. The experimental status of the neutrino masses is not very good, so that we have only upper limits for them:

$$m_{\nu_e} \leqslant 60\,\mathrm{eV}, \qquad m_{\nu_\mu} \leqslant 1.2\,\mathrm{MeV} \tag{5.1}$$

Our numerical results are given in Table 1. Although we considered here only the lowest-order Feynman graph, inclusion of higher orders requires merely small modifications for l_{ph} and $l_{e\mu N}$. The value of the electron self-energy does not change for $m_\nu \lesssim 1$ MeV.

Yet, it is important that the anomalous magnetic moments of the electron and the muon can consistently be explained. From the point of view of the successful unification of the electromagnetic and weak interaction, a unification of the neutrino and the electron must be significant.

5.2. The Anomalous Magnetic Moments of the Electron and the Muon. The fantastic measurement of the g factor anomaly for μ^+, $a \equiv (g - 2)/2 = (1\,165\,895 \pm 27) \times 10^{-9}$, were recently done by using the CERN Muon Storage Ring (Bailey et al., 1975). And it turned out that the experimental value is $(13 \pm 29) \times 10^{-9}$ below the theoretical value in which the pure QED contribution up to sixth-order terms, $(1\,165\,835 \pm 2.2) \times 10^{-9}$ (Combley and Picasso, 1974), and a hadronic contribution of $(73 \pm 10) \times 10^{-9}$ are included. The eighth-order term makes a contribution of $((3.2 \pm 0.2) \sim 5) \times 10^{-9}$ (Lautrup, 1972; Calmet and Peterman, 1975b) and the weak-interaction contribution is conjectured to have about the same magnitude as that (Bailey et al., 1975; Jackiw and Weinberg, 1972). Therefore, if there were a discrepancy between the experimental and theoretical values even after taking proper account of contribution from the weak interaction, it would be evidence for breakdown of the usual theory of QED. For this reason, the g-factor anomaly is extremely important.

Calmet and Peterman (1975a) estimated the pure QED contribution up to sixth-order terms and their result is $(1\,165\,827 \pm 3) \times 10^{-9}$. The hadronic contributions were estimated by several groups, Gourdin–de Rafael (1969), Branion–Etim–Greco (1972), Barger–Long–Olsson (1975), and Calmet–Narison–Perrottet–de Rafael (1976), and their results are $(66 \pm 9) \times 10^{-9}$, $(68 \pm 9) \times 10^{-9}$, $(66 \pm 10) \times 10^{-9}$, and $(66.4 \pm 10.2) \times 10^{-9}$, respectively.

With Calmet–Peterman's values (1 165 827 ± 3) × 10^{-9} for the pure QED contribution up to sixth-order terms plus (3.2 ± 0.2) × 10^{-9} for the eighth-order term and Barger–Long–Olsson's value for the hadronic contribution, we obtain $a_{\text{exp}} - (a_{\text{QED}} + a_{\text{had}}) = -1 \times 10^{-9}$. Accordingly, without weak-interaction contribution, the experimental value of the μ^+ g-factor anomaly is smaller than theoretical value by (1 ~ 16) × 10^{-9}. If weak-interaction contribution has a positive sign, the discrepancy becomes larger. Continuing measurement by CERN Muon Storage Ring Collaboration (Bailey et al., 1975) will reduce the error.

On the other hand, the g-factor anomaly for the electron was measured by two groups (Wesley and Rich, 1971; Granger and Ford, 1972; Walls and Stein, 1973) since 1970. The result obtained by Wesley and Rich (1971) is (1 159 656.7 ± 3.5) × 10^{-9}, which is larger than the theoretical values obtained by several people: 1.7 × 10^{-9} (Levine and Wright, 1971), 3.8 × 10^{-9} (Kinoshita and Cvitanovic, 1972), 4.8 × 10^{-9} (Levine and Wright, 1973), and 5.0 × 10^{-9} (Cvitanovic and Kinoshita, 1974).

The anomalous magnetic moments of the electron and the muon obtained in the present theory have a small deviation Δa from the Schwinger term. If this deviation is of order of 10^{-9}, our corrections to the higher-order terms can be neglected. The numerical results are given in Table 1. Since our correction is a function of $\tilde{m}_-$, the results are different for the cases of electron and muon. However, it should be noticed that the correction to the muon $(g - 2)$ factor has an opposite sign to that of the electron. At present, our corrections to the $(g - 2)$ factor anomaly have correct signs.

6. CONCLUDING REMARKS

It has been shown that our theory gives correct signs and magnitudes for the neutrino–electron mass difference, except the neutron–proton mass difference, and the $(g - 2)$ factor anomaly of the electron and the muon. It should also be emphasized that our theory is divergence free. From numerical calculations of various physical quantities, we conclude that the fundamental lengths contained in the theory are

$$l_{\text{ph}} = \frac{\hbar}{\Lambda c} = 2 \times 10^{-16}\,\text{cm} \qquad (6.1)$$

for the photon and

$$l_{e\mu N} = \frac{\hbar}{c(\tilde{m}_+{}^2 + \tilde{m}_-{}^2)^{1/2}} \simeq 0.25 \times 10^{-19}\,\text{cm} \qquad (6.2)$$

for the fermion. The fundamental length (6.1) corresponds to $\Lambda \simeq 100$ GeV and, thus, the critical mass (Cheon, 1978) is ≃50 GeV, which seems to be a reasonable magnitude from the view of the intermediate vector beson (Salam, 1968; Weinberg, 1967).

In the present paper, we have given discussions on the low-energy quantities. Since there is a possibility that a significant deviation from the usual QED may appear in the region of large momentum transfer, it would be interesting to test our theory for the high-energy scattering such as $e^+e^- \to e^+e^-$, and $e^+e^- \to \mu^+\mu^-$, etc.

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APPENDIX

We shall give the formulas which are frequently used in calculation of the self-energy of the electron, the vacuum polarization, and the vertex correction.

First of all, putting $x = k_0$ and $a^2 = \mathfrak{k}^2 + L$ in the integral formula

$$\int_{-\infty}^{\infty} \frac{dx}{x^2 - a^2 + i\epsilon} = -\frac{i\pi}{a} \tag{A.1}$$

we obtain the following expression:

$$\int_{-\infty}^{\infty} \frac{dk_0}{k_0{}^2 - (\mathfrak{k}^2 + L) + i\epsilon} = \frac{-i\pi}{(\mathfrak{k}^2 + L)^{1/2}} \tag{A.2}$$

Differentiating (A.2) with respect to L gives

$$\int_{-\infty}^{\infty} \frac{dk_0}{[k_0{}^2 - (\mathfrak{k}^2 + L) + i\epsilon]^2} = \frac{i\pi}{2(\mathfrak{k}^2 + L)^{3/2}} \tag{A.3}$$

Differentiating (A.3) again with respect to L and considering

$$\int_{-\infty}^{\infty} \frac{d^3k}{(\mathfrak{k}^2 + L)^{5/2}} = \frac{4\pi}{3L} \tag{A.4}$$

we obtain

$$\int_{-\infty}^{\infty} \frac{d^4k}{(k^2 - L + i\epsilon)^3} = -\frac{i\pi^2}{2L} \tag{A.5}$$

Since $k^2 - 2pk + \Delta = (k - p)^2 + (\Delta - p^2)$, the following integral can be carried out with the help of the formula (A.5):

$$\int_{-\infty}^{\infty} \frac{(1; k_\sigma)d^4k}{(k^2 - 2pk + \Delta + i\epsilon)^3} = \int_{-\infty}^{\infty} \frac{(1; \bar{k}_\sigma + p_\sigma)d^4\bar{k}}{(\bar{k}^2 - p^2 + \Delta + i\epsilon)^3}$$
$$= -\frac{i\pi^2(1; p_\sigma)}{2(p^2 - \Delta)} \tag{A.6}$$

where $\sigma = 0, 1, 2, 3$ and we used the fact that the integral of odd function with respect to $\bar{k}$ vanishes.

Making use of the formula

$$\frac{1}{a^2 b} = \int_0^1 \frac{2x\,dx}{[ax + b(1-x)]^3} \tag{A.7}$$

we find

$$\int_{-\infty}^{\infty} \frac{(1; k_\sigma) d^4 k_\sigma}{(k^2 - 2p_1 k + \Delta_1)^2 (k^2 - 2p_2 k + \Delta_2)}$$

$$= \int_{-\infty}^{\infty} d^4 k \int_0^1 dx \frac{2x(1; k_\sigma)}{(k^2 - 2p_x k + \Delta_x)^3}$$

$$= \int_0^1 \frac{i\pi^2 x(1; p_{x\sigma})}{(\Delta_x - p_x^2)} dx \tag{A.8}$$

where

$$p_x = x p_1 + (1-x) p_2 \tag{A.9}$$

$$\Delta_x = x\Delta_1 + (1-x)\Delta_2$$

and we dropped the factor $i\epsilon$ in the denominators. By differentiating both sides of equation (A.8) with respect to Δ_2, one can easily find

$$\int_{-\infty}^{\infty} \frac{(1; k_\sigma) d^4 k}{(k^2 - 2p_1 k + \Delta_1)^2 (k^2 - 2p_2 k + \Delta_2)^2} = i\pi^2 \int_0^1 \frac{(1; p_{x\sigma}) x(1-x)}{(\Delta_x - p_x^2)^2} dx \tag{A.10}$$

Differentiating of (A.8) with respect to p_{20} for $\sigma = 0$ and p_{2j} for $\sigma = j$ yields

$$\int_{-\infty}^{\infty} \frac{k_\sigma k_\tau d^4 k}{(k^2 - 2p_1 k + \Delta_1)^2 (k^2 - 2p_2 k + \Delta_2)^2}$$

$$= i\pi^2 \int_0^1 \frac{x(1-x)[p_{x\sigma} p_{x\tau} + (1/2) g_{\sigma\tau}(\Delta_x - p_x^2)]}{(\Delta_x - p_x^2)^2} dx \tag{A.11}$$

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